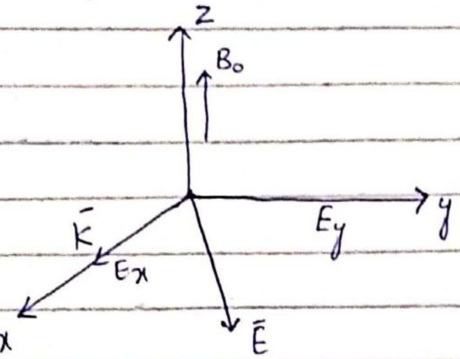
6-3-19. Wednesday.

Extraordinary waves (mode) :- $\bar{E}_1 \perp \bar{B}_0$

Suppose B_0 is along z-axis, k is along x-axis & $\bar{E}$ is in xy-plane.
We study both longitudinal & transverse wave in Extraordinary wave. It is having both parallel & perpendicular propagation.

$$\bar{E} = E_x\hat{x} + E_y\hat{y}$$
$$k = k\hat{x}$$
$$B = B_0\hat{z}$$



Eq. of motion in linearized form,

$$m n_0 \frac{\partial \bar{V}_1}{\partial t} = -n_0 e[\bar{E}_1 + \bar{V}_1 \times \bar{B}_0] \quad (1)$$

$$\bar{V}_1 = \frac{-ie}{m\omega}[E_1 + \bar{V}_1 \times B_0] \quad (2)$$

X-component:

$$V_x = \frac{-ie}{m\omega}(E_x + V_y B_0) \quad (3)$$

∵ Ignoring the subscript 1 for simplicity.

Y-component:

$$V_y = \frac{-ie}{m\omega}(E_y - V_x B_0) \quad (4)$$

Substituting the Value of V_y from eq.(4) in eq.(3)

$$V_x = \frac{-ie}{m\omega}E_x - \frac{ieB_0}{m\omega}\left[\frac{-ie}{m\omega}(E_y - V_x B_0)\right]$$

$$V_x = \frac{-ie}{m\omega}E_x - \frac{e^2B_0}{m^2\omega^2}E_y + \frac{e^2B_0^2}{m^2\omega^2}V_x$$

$$V_x = \frac{-e}{m\omega}\left(iE_x + \frac{eB_0}{m\omega}E_y\right) + \frac{e^2B_0^2}{m\omega}V_x)$$

$$V_x = -\frac{e}{m\omega}\left(iE_x + \frac{eB_0}{m\omega}E_y\right) + \frac{\omega_c^2}{\omega^2}V_x$$

$$V_x\left(1 - \frac{\omega_c^2}{\omega_e^2}\right) = \frac{-e}{m\omega}\left(iE_x + \frac{\omega_c}{\omega}E_y\right)$$

$$V_x = -\frac{e}{m\omega}\left(iE_x + \frac{\omega_c}{\omega}E_y\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} \quad (5)$$

Now, Eq # 4 becomes

$$V_y = -\frac{ie}{m\omega}E_y + i\frac{\omega_c}{\omega}V_x \quad (6)$$

Substituting eq.(3) in eq.(6)

$$V_y = -\frac{ie}{m\omega}E_y + \frac{i\omega_c}{\omega}\left[-\frac{ie}{m\omega}\left(E_x + V_y B_o\right)\right].$$

$$V_y = -\frac{ie}{m\omega}E_y + \frac{\omega_c e}{m\omega^2}E_x + \frac{\omega_c^2}{\omega^2}V_y$$

$$V_y = \frac{e}{m\omega}\left(-iE_y + \frac{\omega_c}{\omega}E_x\right) + \frac{\omega_c^2}{\omega^2}V_y.$$

$$V_y = \frac{e}{m\omega}\left(-iE_y + \frac{\omega_c}{\omega}E_x\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} \quad (7)$$

From Maxwell's equation & after doing simple mathematics, we can write

$$(\omega^2 - c^2k^2)\bar{E} + c^2k(\bar{k}\cdot\bar{E}) = i4\pi n_o e\bar{V}_1 \quad (A)$$

X- component:

$$(\omega^2 - c^2\cancel{k^2})E_x + \cancel{c^2k}(k\cdot E_x) = i4\pi n_o e V_x$$

$$\omega^2 E_x = i4\pi n_o e V_x\omega \quad (8)$$

Y-component:

$$(\omega^2 - c^2k^2)E_y + c^2k\,(0) = i4\pi n_o e V_y.$$

$$(\omega^2 - c^2k^2)E_y = i4\pi n_o e V_y\omega \quad (9)$$

$(K\cdot E) = 0$ because k and E_y are in opposite direction.

Substituting the values of V_x & V_y from eq.(5) & (7) in eq.(8) & (9) respectively.

$$\omega^2 E_x = i4\pi n_o e\omega\left(-\frac{e}{m\omega}\left(iE_x + \frac{\omega_c}{\omega}E_y\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}\right) \quad (10)$$

$$(\omega^2 - c^2k^2)E_y = i4\pi n_o e\omega\left(\frac{e}{m\omega}\left(-iE_y + \frac{\omega_c}{\omega}E_x\right)\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}\right) \quad (11)$$

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$$+\frac{c^2k^2}{\omega^2}(\omega^2-\omega_c^2) = \omega^2-\omega_h^2 - \frac{\omega_{pe}^4\omega_c^2/\omega^2}{(\omega^2-\omega_h^2)}$$

$$\frac{c^2k^2}{\omega^2} = \frac{(\omega^2-\omega_h^2)^2 - \omega_{pe}^4\omega_c^2/\omega^2}{(\omega^2-\omega_c^2)(\omega^2-\omega_h^2)}$$

$$\frac{c^2k^2}{\omega^2} = \frac{(\omega^2-\omega_h^2)(\omega^2-\omega_c^2-\omega_{pe}^2) - \omega_{pe}^4\omega_c^2/\omega^2}{(\omega^2-\omega_c^2)(\omega^2-\omega_h^2)}$$

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{pe}^2(\omega^2-\omega_h^2) + \omega_{pe}^4\omega_c^2/\omega^2}{(\omega^2-\omega_c^2)(\omega^2-\omega_h^2)}$$

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{pe}^2(\omega^2-\omega_{pe}^2-\omega_c^2) + \omega_{pe}^4\omega_c^2/\omega^2}{(\omega^2-\omega_c^2)(\omega^2-\omega_h^2)}$$

Dividing & multiplying ω_{pe}^2 by ω^2.

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\frac{\omega_{pe}^2}{\omega^2}\omega^2(\omega^2-\omega_{pe}^2-\omega_c^2) + \omega_{pe}^4\omega_c^2/\omega^2}{(\omega^2-\omega_c^2)(\omega^2-\omega_h^2)}$$

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\frac{\omega_{pe}^2}{\omega^2}[\omega^2(\omega^2-\omega_{pe}^2) - \omega^2\omega_c^2] + \omega_{pe}^4\omega_c^2/\omega^2}{(\omega^2-\omega_c^2)(\omega^2-\omega_h^2)}$$

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\frac{\omega_{pe}^2}{\omega^2}[\omega^2(\omega^2-\omega_{pe}^2) - \omega_c^2(\omega^2-\omega_{pe}^2)]}{(\omega^2-\omega_c^2)(\omega^2-\omega_h^2)}$$

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\frac{\omega_{pe}^2}{\omega^2}[\cancel{(\omega^2-\omega_c^2)}(\omega^2-\omega_{pe}^2)]}{\cancel{(\omega^2-\omega_c^2)}(\omega^2-\omega_h^2)}$$

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}\,\frac{\omega^2-\omega_{pe}^2}{\omega^2-\omega_h^2}$$

This is the dispersion relation of Extraordinary mode. This is partially electrostatic & electromagnetic mode.

$\frac{c^2k^2}{\omega^2}$ = Refractive index.

$$\frac{c^2}{\omega^2/k^2} = \frac{c^2}{V_\phi^2} = \text{Ratio of Speed of light to the ratio of phase velocity.}$$

There are two situations of wave.

i) Cut-off

ii) Resonance.

In plasma, if we said wave is cut-off then k is zero $(k=0)$

$$k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k} = \frac{2\pi}{0} \quad \text{then } \lambda = \infty$$

Resonance:-

It means $k = \infty$ then

$$\lambda = \frac{2\pi}{\infty} = 0$$

If wave hase infinite k & wavelength is zero then it is said to be resonance.

Resonance of Extraordinary mode:-

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}\,\frac{\omega^2 - \omega_{pe}^2}{\omega^2 - \omega_h^2}$$

$$c^2k^2 = \omega^2 - \frac{\omega_{pe}^2(\omega^2 - \omega_{pe}^2)}{\omega^2 - \omega_h^2} \quad (\because \text{multiplying by } \omega^2).$$

Taking inverse,

$$\frac{1}{c^2k^2} = \frac{\omega^2 - \omega_h^2}{\omega^2(\omega^2 - \omega_h^2) - \omega_{pe}^2(\omega^2 - \omega_{pe}^2)}$$

$$\frac{1}{c^2(\infty)} = \frac{\omega^2 - \omega_h^2}{\omega^2(\omega^2 - \omega_h^2) - \omega_{pe}^2(\omega^2 - \omega_{pe}^2)}$$

$$0 = \frac{\omega^2 - \omega_h^2}{\omega^2(\omega^2 - \omega_h^2) - \omega_{pe}^2(\omega^2 - \omega_{pe}^2)}$$

$$\omega^2 - \omega_h^2 = 0$$

$$\omega^2 = \omega_{pe}^2 + \omega_c^2$$

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Cut off :-

It means $k = 0$

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}\,\frac{\omega^2 - \omega_{pe}^2}{\omega^2 - \omega_h^2}$$

$$1 = \frac{\omega_{pe}^2}{\omega^2}\,\frac{\omega^2 - \omega_{pe}^2}{\omega^2 - \omega_h^2}$$

$$1 = \frac{\omega_{pe}^2}{\omega^2}\,\frac{\omega^2 - \omega_{pe}^2}{(\omega^2 - \omega_{pe}^2 - \omega_c^2)}$$

$$1 = \frac{\omega_{pe}^2}{\omega^2}\cdot\frac{1}{1 - \frac{\omega_c^2}{\omega^2 - \omega_{pe}^2}}$$

$$1 - \frac{\omega_c^2}{\omega^2 - \omega_{pe}^2} = \frac{\omega_{pe}^2}{\omega^2}$$

$$1 - \frac{\omega_{pe}^2}{\omega^2} = \frac{\omega_c^2}{\omega^2 - \omega_{pe}^2}$$

$$1 - \frac{\omega_{pe}^2}{\omega^2} = \frac{\omega_c^2}{\omega^2\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)}$$

$$\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)^2 = \frac{\omega_c^2}{\omega^2}$$

$$1 - \frac{\omega_{pe}^2}{\omega^2} = \pm\frac{\omega_c}{\omega}$$

$\omega^2 \mp \omega\omega_c - \omega_{pe}^2 = 0$ ∵ multiply by ω

This is quadratic equation:

$\omega^2 - \omega\omega_c - \omega_{pe}^2 = 0$; $\omega^2 + \omega\omega_c - \omega_{pe}^2 = 0$

$$\omega_R = \frac{1}{2}\left[\omega_c + \sqrt{\omega_c^2 + 4\omega_{pe}^2}\right]$$

$$\omega_L = \frac{1}{2}\left[-\omega_c + \sqrt{\omega_c^2 + 4\omega_{pe}^2}\right]$$

It is left hand cut-off frequency.